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Sparse Identification for Fractional Order Systems

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***Abstract*—This study extends the Sparse Identification of Nonlinear Dynamics (SINDY) algorithm to the discovery of fractional order systems. The SINDY optimization problem was modified to incorporate candidate fractional orders in the Caputo sense, with discretization executed using Grünwald-Letnikov finite differences. Identification of the governing equations was done separately for each state. Noise was handled through weak formulation, while sparse regression was performed via sequential thresholding. Discovery of the correct fractional orders was done through parameter sweep. The results showed that the algorithm was able to identify the governing equations of the fractional order Lorenz system, but still suffered from its inherent memory effects.**

***Index Terms*—fractional order systems, SINDY, system identification**

I. Introduction

Fractional order systems, also known as non-integer order systems, are systems described by fractional-order differential equations, where the order of differentiation is a non-integer value. These systems exhibit more complex behaviors than classical systems, such as long-term memory and non-locality [1]. Because real systems are not exactly integer order, fractional order models have found applications in many fields, including control engineering, signal processing, physics, and finance. They have also been used to accurately model a wide range of phenomena that are indescribable through integer-order equations including electrochemical processes [2] [3], viscoelasticity [4], biological systems [5] [6], chaotic systems [7], and energy storage devices such as supercapacitors and batteries [8] [9] [10]. Additionally, they can also be used to model even systems composed exclusively of a large number of integer order devices [11].

As noted in [12], the data-driven identification of unknown FO systems is significantly more complicated compared to the identification of integer order systems because the former has an additional step of choosing the correct order for the fractional operators. Identification techniques that rely on time-domain methods [13], continuous-order distributions [12], and genetic algorithms [14] have been developed for this purpose. However, it appears that the use of sparsity promoting techniques has not yet been explored.

In 2016, the Sparse Identification of Nonlinear Dynamics or SINDY [15] has been proposed for nonlinear system identification. A machine learning algorithm, it is used to find generalizable and interpretable mathematical models of complex systems directly from observational data. The SINDY algorithm starts with a library of candidate functions, which can be chosen to include a variety of nonlinear functions such as polynomials and trigonometric functions. Through sparse optimization, SINDY identifies which among these functions are the most important in describing the system's dynamics. The algorithm and its derivatives have been widely applied in a variety of scientific and engineering domains, and has been shown to be effective in accurately identifying complex nonlinear dynamics in a parsimonious manner [15] [16] [17] [18].

Despite this, SINDY assumes that the state equations are first order by default and therefore, it cannot be used in the discovery of fractional equations involving unknown orders without modifying the original algorithm. This study is thus aimed at extending the SINDY algorithm to fractional order systems.

II. Preliminaries

A. Mathematics of the Sparse Identification of Nonlinear Dynamics and Weak Formulation

In SINDY [15], the underlying dynamics of a system is represented as a first order system of differential equations in the form

$$\dot{\mathbf{x}}(t) = \mathbf{f}(\mathbf{x}(t)) \tag{1}$$

where $\mathbf{x}(t) \in \mathbb{R}^n$ is the state at time t, and $\mathbf{f}(\mathbf{x}(t))$ are the functions that define the dynamics. To identify the governing equations, the sparse regression problem

$$\dot{\mathbf{X}} = \mathbf{\Theta}(\mathbf{X})\mathbf{\Xi} \tag{2}$$

is formulated. In (2), $\mathbf{X}$ is an $m \times n$ matrix made up of measurements of the n states sampled at equidistant points in time $(t_1, t_2, ..., t_m)$, and $\dot{\mathbf{X}}$ which is also an $m \times n$ matrix, is composed of the first derivatives of $\mathbf{X}$. Note here that $m \gg n$ because there are usually more samples than states.

Meanwhile, $\mathbf{\Theta}(\mathbf{X})$ is an $m \times j$ data matrix composed of candidate nonlinear functions θ_q of the columns of $\mathbf{X}$:

$$\mathbf{\Theta}(\mathbf{X}) = \begin{bmatrix} \theta_1(\mathbf{X}) & \theta_2(\mathbf{X}) & \dots & \theta_j(\mathbf{X}) \end{bmatrix} \tag{3}$$

The candidate nonlinear functions θ_q are chosen carefully depending on the application and could be any mathematical function e.g. polynomials and trigonometric functions. Lastly, $\mathbf{\Xi}$ is made up of the coefficient column vectors $\xi_k \in \mathbb{R}^j$

that are calculated using a sparse regression algorithm such as the Sequentially Thresholded Least Squares introduced in [15]. The nonzero coefficients of $\boldsymbol{\xi}_k \in \mathbb{R}^j$s determine which columns in $\boldsymbol{\Theta}(\mathbf{X})$ are active in the dynamics for the kth state equation in (1). That is, after $\boldsymbol{\Xi}$ is obtained, the kth governing equation is written as

$$\dot{x}_k = \boldsymbol{\Theta}(\mathbf{x})\boldsymbol{\xi}_k \tag{4}$$

where $\boldsymbol{\Theta}(\mathbf{x})$ is a row vector of symbolic functions that represent the columns of $\boldsymbol{\Theta}(\mathbf{X})$.

One important factor to consider when applying SINDY is that when directly calculating derivatives from measurement data, any noise present in the true signal is amplified. Moreover, attempting to eliminate noise before differentiation often yields unsatisfactory outcomes. In [15], this issue was circumvented by calculating the derivative through total variation regularization [19]. Reference [20] adopted a better approach by performing sparse regression on the resulting equation after both sides of (2) are integrated which effectively averaged out the noise. To elaborate, this method entails the transformation of (2) into

$$\dot{\mathbf{X}}^I = \boldsymbol{\Theta}^I(\mathbf{X})\boldsymbol{\Xi}^I \tag{5}$$

where

$$\dot{\mathbf{X}}^I = \begin{bmatrix} x_1(t_1) \\ \vdots \\ x_1(t_m) \\ \vdots \\ x_n(t_1) \\ \vdots \\ x_n(t_m) \end{bmatrix} - \begin{bmatrix} x_1(t_1) \\ \vdots \\ x_1(t_1) \\ \vdots \\ x_n(t_1) \\ \vdots \\ x_n(t_1) \end{bmatrix}, \tag{6}$$

$$\boldsymbol{\Theta}^I(\mathbf{X}) = \begin{bmatrix} \boldsymbol{\Omega}(\mathbf{X}) & 0 & \dots & 0 \\ 0 & \boldsymbol{\Omega}(\mathbf{X}) & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \boldsymbol{\Omega}(\mathbf{X}) \end{bmatrix}, \tag{7}$$

and

$$\boldsymbol{\Xi}^I = \begin{bmatrix} \boldsymbol{\xi}_1^T & \boldsymbol{\xi}_1^T & \dots & \boldsymbol{\xi}_n^T \end{bmatrix}^T \tag{8}$$

with the superscript T representing the transpose operation. Lastly in (7),

$$\boldsymbol{\Omega}(\mathbf{X}) = \begin{bmatrix} \int_{t_1}^{t_1} \boldsymbol{\theta}_1(\mathbf{x}(\tau))\, d\tau & \dots & \int_{t_1}^{t_1} \boldsymbol{\theta}_j(\mathbf{x}(\tau))\, d\tau \\ \vdots & \ddots & \vdots \\ \int_{t_1}^{t_m} \boldsymbol{\theta}_1(\mathbf{x}(\tau))\, d\tau & \dots & \int_{t_1}^{t_m} \boldsymbol{\theta}_j(\mathbf{x}(\tau))\, d\tau \end{bmatrix} \tag{9}$$

Note that this is equivalent to just integrating both $\dot{\mathbf{X}}$ and the columns of $\boldsymbol{\Theta}(\mathbf{X})$ without stacking the states and coefficient vectors, but was written this way for a neater presentation, and to reflect the discussion in [20].

B. Fractional Differential Equations with Nonhomogeneous Initial Values

There are several definitions of the fractional derivative, with the Riemann-Liouville (RL), Grünwald-Letnikov (GL), and Caputo definitions being among the most studied. References [1] and [21] present a comprehensive discussion of these definitions and their relationships.

Let $D^\alpha f(t)$ denote the αth order derivative of a function $f(t)$ when $\alpha > 0$ or the αth repeated integral when $\alpha < 0$. When $\alpha = 0$, $D^\alpha f(t) = f(t)$. From here onwards, focus will be on the case when $0 < \alpha < 1$ [22]. The time interval for all fractional operators will also be set to $[0, t]$ for simplicity. Fractional integrals will be sometimes referred to as fractional derivatives unless there is a need for distinction between the two because of the aforementioned properties.

For real physical systems described by fractional differential equations, the Caputo operator is suitable to use because it produces meaningful results for initial value problems, and because the initial values are more likely to be nonhomogeneous. The Caputo operator ${}^{C}_{0}D^{\alpha}_{t}$ is related to the RL operator ${}^{RL}_{0}D^{\alpha}_{t}$ by the following:

$${}^{C}_{0}D^{\alpha}_{t}\, f(t) = {}^{RL}_{0}D^{\alpha}_{t}\, f(t) - r^{\alpha}_{0}(t) f(0) \tag{10}$$

where

$$r^{\alpha}_{0}(t) = \frac{t^{-\alpha}}{\Gamma(1-\alpha)} \tag{11}$$

which is a correction term. Since the GL definition is equivalent to the RL definition [1], it can be used to approximate the Caputo derivative. The GL differintegral, denoted by ${}^{GL}_{0}D^{\alpha}_{t}\, f(t)$ is given by

$${}^{GL}_{0}D^{\alpha}_{t}\, f(t) = \lim_{h\to 0} \frac{1}{h^\alpha} \sum_{j=0}^{\lfloor \frac{t}{h} \rfloor} (-1)^j \binom{\alpha}{j} f(t - jh) \tag{12}$$

where the subscripts 0 and t denote the time interval, $\lfloor \cdot \rfloor$ is the floor operation, and $\binom{\alpha}{j}$ are binomial coefficients obtained using the gamma function:

$$\binom{\alpha}{j} = \frac{\Gamma(\alpha+1)}{\Gamma(j+1)\Gamma(\alpha-j+1)} \tag{13}$$

Note that the expression for the binomial coefficients in (13) is responsible for the operator's dependence on the history of $f(t)$, or its nonlocal characteristic. This distinguishes fractional from integer calculus.

As mentioned, (12) is equivalent to the RL derivative. If h is taken to be very small, the limit can be disregarded and (12) becomes the GL approximation using finite differences denoted by ${}^{GL}_{0}\Delta^{\alpha}_{t}\, f(t)$, i.e.

$$\begin{aligned} {}^{RL}_{0}D^{\alpha}_{t}\, f(t) = {}^{GL}_{0}D^{\alpha}_{t}\, f(t) &\approx {}^{GL}_{0}\Delta^{\alpha}_{t}\, f(t) \\ &= \frac{1}{h^\alpha} \sum_{j=0}^{\lfloor \frac{t}{h} \rfloor} (-1)^j \binom{\alpha}{j} f(t - jh) \end{aligned} \tag{14}$$

In practice, (14) is seldom used directly to find the derivative because evaluating the gamma function repeatedly is slow. A recursion is used instead:

$$ {}^{GL}_{0}\Delta_t^{\alpha} f(t) = \frac{1}{h^{\alpha}} \sum_{j=0}^{\lfloor \frac{t}{h} \rfloor} (-1)^j w_j f(t-jh) \tag{15} $$

where the coefficients

$$ w_j = \left(1 - \frac{\alpha+1}{j}\right) w_{j-1} \tag{16} $$

with

$$ w_0 = 1 \tag{17} $$

It must be noted that for sequential fractional order operators, the commutative law generally does not hold [23]. One special case is that when β is an integer, the following is true:

$$ \begin{aligned} {}^{RL}_{0}D_t^{\beta} \left[{}^{RL}_{0}D_t^{\alpha}\right] f(t) &= \frac{d^{\beta}}{dt^{\beta}} \left[{}^{RL}_{0}D_t^{\alpha}\right] f(t) \\ &= {}^{RL}_{0}D_t^{\beta+\alpha} f(t) \end{aligned} \tag{18} $$

III. Sparse Identification for Fractional Order Systems (SIFOS)

To incorporate fractional order derivatives, the SINDY problem (2) is reformulated into

$$ \begin{bmatrix} {}^{C}_{0}D_t^{\alpha_1}\mathbf{x}_1 & \dots & {}^{C}_{0}D_t^{\alpha_n}\mathbf{x}_n \end{bmatrix} = \mathbf{\Theta}(\mathbf{X})\mathbf{\Xi} \tag{19} $$

where the orders of differentiation $\alpha_k \in (0,1)$. This is in the form of a fractional order state space model discussed in [22]. Using relations (10) and (14) and then integrating both sides [20] by applying property (18) with $\beta = -1$,

$$ \begin{aligned} &\begin{bmatrix} {}^{GL}_{0}\Delta_t^{\alpha_1}\mathbf{x}_1 & \dots & {}^{GL}_{0}\Delta_t^{\alpha_n}\mathbf{x}_n \end{bmatrix} - \\ &\begin{bmatrix} \mathbf{r}_0^{-1+\alpha_1}x_1(0) & \dots & \mathbf{r}_0^{-1+\alpha_n}x_n(0) \end{bmatrix} = \mathbf{\Omega}(\mathbf{X})\mathbf{\Xi} \end{aligned} \tag{20} $$

where $\mathbf{\Omega}(\mathbf{X})$ is the same matrix of the integrals of candidate nonlinear functions in (9), while the second term is composed of the correction terms due to the initial conditions, i.e.

$$ \mathbf{r}_0^{-1+\alpha_k} = \begin{bmatrix} r_0^{-1+\alpha_k}(t_1) & \dots & r_0^{-1+\alpha_k}(t_m) \end{bmatrix}^T \tag{21} $$

Formally, the sparse regression problem for the kth state can be stated as

$$ \begin{aligned} (\alpha_k, \boldsymbol{\xi}_k) = \underset{(\hat{\alpha}_k, \hat{\boldsymbol{\xi}}_k)}{\operatorname{argmin}} \| {}^{C}_{0}D_t^{-1+\hat{\alpha}_k}\mathbf{x}_k - \mathbf{\Omega}(\mathbf{X})\hat{\boldsymbol{\xi}}_k \|_2^2 \\ + \gamma \| \hat{\boldsymbol{\xi}}_k \|_1 \end{aligned} \tag{22} $$

where $\|\cdot\|_2^2$ and $|\cdot|$ represent the squared L2 norm and L1 norm, respectively, while γ is a multiplier that penalizes the complexity of $\boldsymbol{\xi}_k$.

Note that it is possible and easier to identify the governing equations for each state independently of the other states. For the kth state,

$$ \begin{bmatrix} {}^{GL}_{0}\Delta_t^{\alpha_k}\mathbf{x}_k - \mathbf{r}_0^{-1+\alpha_k}x_k(0) \end{bmatrix} = \mathbf{\Omega}(\mathbf{X})\boldsymbol{\xi}_k \tag{23} $$

Equation (23) allows candidate fractional orders to be included in the regression problem. Hence, during the identification process, columns corresponding to candidate values for α_k can be added on the lefthand side, with corresponding additional sparse column vectors on the righthand side, i.e., utilizing the Caputo operator for brevity,

$$ \begin{aligned} &\begin{bmatrix} {}^{C}_{0}D_t^{\alpha_{k1}}\mathbf{x}_k & {}^{C}_{0}D_t^{\alpha_{k2}}\mathbf{x}_k & \dots \end{bmatrix} \\ &= \mathbf{\Omega}(\mathbf{X}) \begin{bmatrix} \boldsymbol{\xi}_{k1} & \boldsymbol{\xi}_{k2} & \dots \end{bmatrix} \end{aligned} \tag{24} $$

To identify the system model, STLS [15] is used to solve for the $\boldsymbol{\xi}_k$s in (24) simultaneously. The idea behind this is that assuming that the governing equations of the system are sparse and that all of the active terms were able to be included in the library matrix, it is expected that the algorithm will converge to a solution where the columns corresponding to the candidate fractional orders that are not close to the true order will not be sparse. Utilizing new candidate α_ks that are closer to the orders of the sparse columns during the previous thresholding, the regression process is repeated until the algorithm results in coefficient vectors that meet the required sparsity.

Throughout this process, each set of sparse columns can be assessed in terms of the fidelity of the corresponding model. If the required level of accuracy is achieved, the governing equation for the kth state is given by the most accurate coefficient vector, and the algorithm is subsequently applied to the other states. In cases where the required accuracy is not met, the thresholding process is repeated.

IV. Results and Discussion

The method was tested on the fractional order Lorenz system given by (25) with initial conditions $(x_1(0), x_2(0), x_3(0)) = (-5, 1, 20)$. This well-known atmospheric model exhibits chaos whose self-intersecting trajectories present difficulties for identification techniques. Scilab scripts in [24] were used to generate the state measurements, which were then corrupted with 2% noise. The simulation was done over $t \in [0, 5]$ at 0.001 t-step.

$$ \begin{cases} D^{0.995}x_1 = 10(x_2 - x_1) \\ D^{0.995}x_2 = x_1(28 - x_3) - x_2 \\ D^{0.995}x_3 = x_1x_2 - \frac{8}{3}x_3 \end{cases} \tag{25} $$

The identified equations using SIFOS were:

$$ \begin{cases} D^{0.995}x_1 = 10.120x_2 - 10.145x_1 \\ D^{0.995}x_2 = 28.150x_1 - 1.085x_2 - 1.001x_1x_3 \\ D^{0.995}x_3 = -2.662x_3 + 0.997x_1x_2 \end{cases} \tag{26} $$

which demonstrates that the method was able to identify the active terms in the fractional order dynamics.

Fig. 1 shows the resulting trajectory of the SIFOS model. For comparison, the trajectory of the recovered model using $\alpha_1 = \alpha_2 = \alpha_3 = 1$ was also plotted; this model was theoretically equivalent to the model recovered using the original SINDY with integral formulation [15] [20]. Both trajectories were in close agreement with the dataset until around $t = 2.5$; however, the SINDY model diverged significantly after that. Meanwhile, the SIFOS model was much closer to the dataset in general during the entire simulation, albeit with error that increased for larger t. This error could be attributed to the

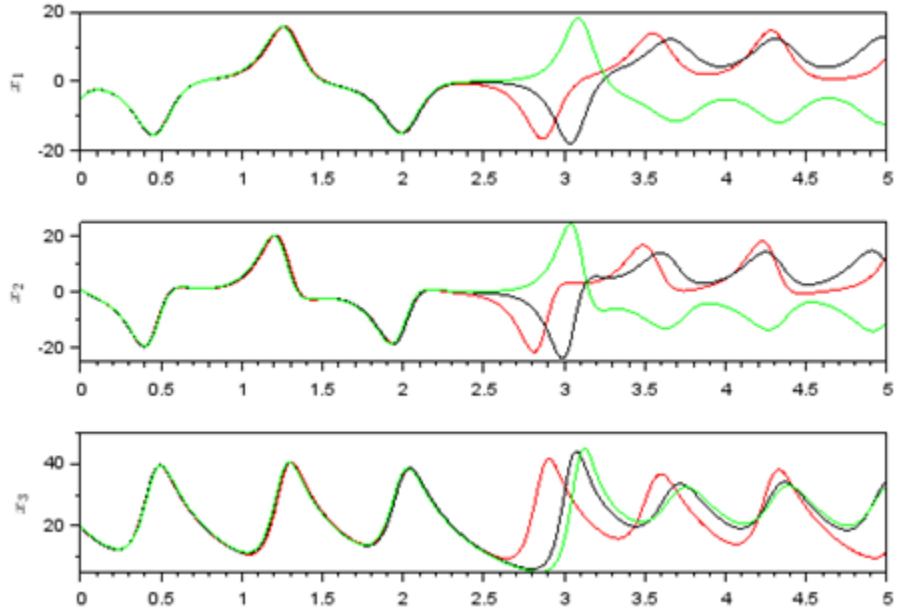

Fig. 1. Trajectories of the Lorenz dataset (red), the recovered model using SINDY (green), and the recovered fractional order model using SIFOS (black). The horizontal axis represents time.

inherent memory effect and sensitivity to initial conditions of the fractional order chaotic system.

V. Conclusion

In this paper, the SINDY algorithm has been extended to include non-integer candidate orders in the optimization process. This method was used to identify the governing equations of the fractional order Lorenz system. The results suggest that sparsity constraints can be a valuable tool for fractional order system identification; however, memory effects should still be taken into consideration when using the recovered model for prediction.

Future studies can be undertaken to improve the accuracy and speed of solving the sparse regression problem. Other aspects of fractional systems that were not covered such as the presence of initialization functions can also be explored.

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References

[1] I. Podlubny, Fractional Differential Equations: An Introduction to Fractional Derivatives, Fractional Differential Equations, to Methods of their Solution and some of their Applications. San Diego, CA: Academic Press, 1999.

[2] V. Martynyuk and M. Ortiguiera, “Fractional order model of an electrochemical capacitor,” in Signal Processing, vol. 107, pp. 355-360, 2015.

[3] X. X. Li, D. Tian, C. H. He, and J. H. He, “A fractal modification of the surface coverage model for an electrochemical arsenic sensor,” in Electrochimica Acta, vol. 296, pp. 491-493, 10 February 2019.

[4] A. Genovese, F. Farroni, and A. Sakhnevych, “Fractional calculus approach to reproduce material viscoelastic behavior, including the time–temperature superposition phenomenon,” in Polymers, vol. 14, no. 20, p. 4412, 2022.

[5] D. Copot, R. De Keyser, J. Juchem, and C. Ionescu, “Fractional order impedance model to estimate glucose concentration: in vitro analysis,” in Acta Polytechnica Hungarica, vol. 14, no. 1, pp. 207-220, 2017.

[6] C. L. MacDonald, N. Bhattacharya, B. P. Sprouse, and G. A. Silva, “Efficient computation of the Grünwald–Letnikov fractional diffusion derivative using adaptive time step memory,” in Journal of Computational Physics, vol. 297, pp. 221-236, 2015.

[7] I. Petras, “Fractional order chaotic systems,” in *Fractional-Order Nonlinear Systems: Modeling, Analysis, and Simulation* (in Nonlinear Physical Science Series). Berlin, Heidelberg: Springer, 2011, pp. 103-184.

[8] T. J. Freeborn, B. Maundy, and A. S. Elwakil, “Fractional-order models of supercapacitors, batteries and fuel cells: a survey,” in Materials for Renewable and Sustainable Energy, vol. 4, no. 9, 2015.

[9] J. A. L. Villanueva, P. R. Iturriaga, and S. R. Bolivar, “A fractional-order model for calendar aging with dynamic storage conditions,” in Journal of Energy Storage, vol. 50, no. 104537, 2022.

[10] X. Sun, J. Ji, B. Ren, G. Chen, and Q. Zhang, “A novel online identification algorithm of lithium-ion battery parameters and model order based on a fractional order model,” in IET Renewable Power Generation, vol. 15, pp. 2396-2408, 2021.

[11] J. T. Machado, and A. Galhano, ”Fractional dynamics: a statistical perspective,” in Journal of Computation and Nonlinear Dynamics, vol. 3, pp. 021201:1–021201:5, 2008.

[12] T. T. Hartley and C. F. Lorenzo, “Fractional-order system identification based on continuous order distributions,” in Signal Processing, vol. 83, no. 11, pp. 2287-2300, 2003.

[13] L. Dorcak, E. A. Gonzalez, J. Terpak, J. Valsa, and L. Pivka, “Identification of fractional-order dynamical systems based on nonlinear function optimization,” in International Journal of Pure and Applied Mathematics, vol. 89, no. 2, pp. 225-250, 2013.

[14] S. Zhou, J. Cao, and Y. Chen, “Genetic algorithm-based identification of fractional-order systems,” in Entropy, vol. 15, pp. 1624-1642, 2013.

[15] S. L. Brunton, J. L. Proctor, and J. N. Kutz, “Discovering governing equations from data by sparse identification of nonlinear dynamical systems,” in Proceedings of the National Academy of Sciences, vol. 113, no. 15, pp. 3932-3937, 2016.

[16] D. R. Gurevich, P. A. Reinbold, and R. O. Grigoriev, “Robust and optimal sparse regression for nonlinear PDE models,” in Chaos, vol. 29, no. 10, p. 103113, 2019.

[17] B. Bhadriraju, M. S. Bangi, A. Narasingam, and J. S. I. Kwon, “Operable adaptive sparse identification of systems: application to chemical processes,”in AIChe Journal, vol. 66, no. 11, p. e16980, 2020.

[18] M. Q. Tran, T. T. Tran, P. H. Nguyen, and G. Pemen, “Sparse identification for model predictive control to support long-term voltage stability,” in IET Generation, Transmission, & Distribution, vol. 17, no. 1, pp. 39-51, 2023.
[19] R. Chartrand, “Numerical differentiation of noisy, nonsmooth data,” in International Scholarly Research Notices, vol. 2011, 2011.
[20] H. Schaeffer and S. G. McCalla, “Sparse model selection via integral terms,” in Physical Review E, vol. 96, no. 2, p. 023302.
[21] R. Scherer, S. L. Kalla, Y. Tang, and J. Huang, “The Grünwald-Letnikov method for fractional differential equations,” in Computers and Mathematics with Applications, vol. 62, no. 3, pp. 902-917, 2011.
[22] L. Dorcak, I. Petras, I. Kostial, and J. Terpak, “Fractional-order state space models,” in Proceedings of the 3rd International Carpathian Control Conference, pp. 193-198, 2002.
[23] D. Xue, Fractional-Order Control Systems: Fundamentals and Numerical Implementations. Berlin: De Gruyter, 2017.
[24] K. Bingi, R. Ibrahim, M. N. Karsiti, S. M. Hassan, V. R. Harindran, “Scilab-based toolbox for fractional-order chaotic systems,” in *Fractional-order Systems and PID Controllers: Using Scilab and Curve Fitting Based Approximation Techniques*, vol. 264, J. Kacprzyk, Ed. Switzerland: Springer Nature, 2020, pp. 227-237.